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Gravity Turn Descent with Quadratic Air Drag

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Introduction

GRAVITY turn guidance has been widely investigated[1–3] and used in practice[4,5] for terminal descent to a planetary surface. The descent method requires that the vehicle thrust vector is aligned opposite to the vehicle velocity vector at all points along the descent trajectory. This may be easily implemented onboard by using the vehicle attitude control system to null body rates about the vehicle velocity vector. The method is also efficient, providing near minimum-fuel descents. Previous studies of the method have assumed that the descent maneuver takes place in a vacuum. However, some scenarios, such as descent to a planetary surface or moon with a significant atmosphere, may lead to more complex descent dynamics.

In this Note the gravity turn maneuver is considered with quadratic air drag. It is assumed that the terminal maneuver is initiated at low altitude, where the air density and vehicle drag coefficient may be considered near constant. Because the gravity turn maneuver requires the vehicle thrust vector to be aligned opposite to the vehicle velocity vector, the angle of attack will also be constant. It is shown that the modified equations of motion have a closed-form solution obtained from Bernoulli's equation.

Vacuum Gravity Turn

First, the standard gravity turn maneuver will be reviewed and the equations of motion solved in closed analytical form. For terminal descent maneuvers, the vehicle planar translational dynamics may be modeled as a point mass moving over a flat surface at altitude h with a uniform local gravitational acceleration g, viz.,

$$\dot{v} = -\alpha g + g \cos\psi \tag{1a}$$

$$v\dot{\psi} = -g \sin\psi \tag{1b}$$

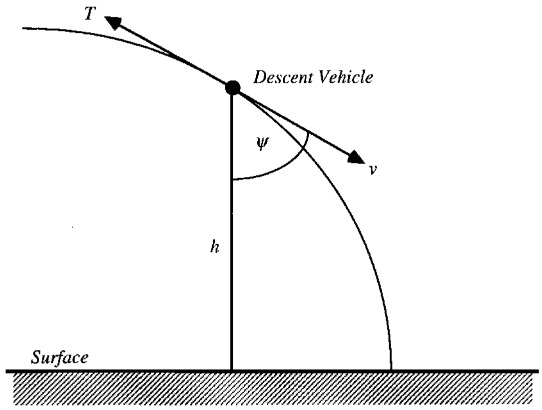

Fig. 1 Schematic gravity turn descent.

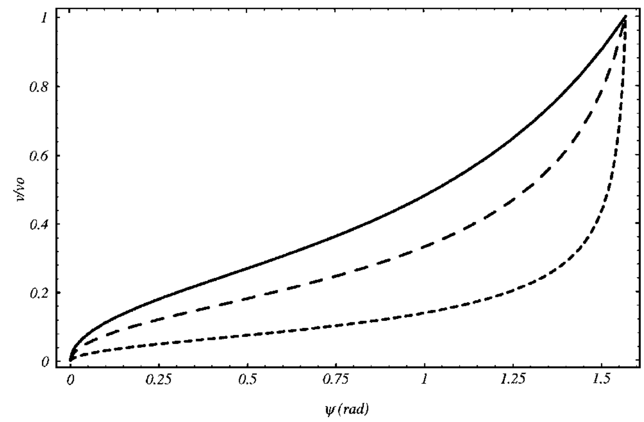

Fig. 2 Velocity-flight-path-angle profile:——, vacuum; – – –, $\beta v_0^2 = 10^{-3}$; and - - - -, $\beta v_0^2 = 10^{-2}$.

where the state variables are illustrated in Fig. 1. The thrust-weight ratio $\alpha = T/mg$ is assumed to be constant during the descent maneuver, although throttling may be used to track defined descent contours.[6]

The preceding equations may now be solved by using the flight-path angle ψ as the independent variable. Then, Eqs. (1) yield a single first-order equation, viz.,

$$\frac{1}{v}\frac{\mathrm{d}v}{\mathrm{d}\psi} = \alpha \operatorname{cosec}\psi - \cot\psi \tag{2}$$

The solution for the vehicle velocity as a function of flight-path angle follows directly by integration and may be written as[1]

$$v(\psi) = v_0 \left[\frac{\cos(\psi/2)}{\cos(\psi_0/2)}\right]^{-(1+\alpha)} \left[\frac{\sin(\psi/2)}{\sin(\psi_0/2)}\right]^{-(1-\alpha)} \tag{3}$$

A gravity turn descent with $\alpha = 1.5$ is shown in Fig. 2. It can be seen that the maneuver has the useful property that a vertical landing is ensured since $v \to 0$ as $\psi \to 0$. To ensure that $v \to 0$ as $h \to 0$ requires an appropriate choice of α for a given initial altitude.[1]

Gravity Turn with Quadratic Air Drag

The gravity turn maneuver is now reconsidered with the addition of quadratic air drag. It will be demonstrated that the equations of motion can again be solved in closed form and that the solutions reduce to the vacuum case derived earlier. Because the thrust vector is oriented opposite the velocity vector, air drag augments the

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thrust-induced braking acceleration. Therefore, including air drag, the equations of motion now become

$$\dot{v} = -(\alpha + \beta v^2)g + g\cos\psi \tag{4a}$$

$$v\dot{\psi} = -g\sin\psi \tag{4b}$$

where

$$\beta = \frac{\rho_* C_D A}{2mg} \tag{5}$$

The drag coefficient C_D, mass m, and aerodynamic reference area A are all assumed to be constant, as is the local atmospheric density ρ_*. Again, from Eqs. (4) the vehicle flight-path angle may be used as the independent variable to obtain a single first-order equation, viz.,

$$\frac{dv}{d\psi} = f_1(\psi)v + f_2(\psi)v^3 \tag{6}$$

where

$$f_1(\psi) = \alpha\,\mathrm{cosec}\,\psi - \cot\psi \tag{7a}$$

$$f_2(\psi) = \beta\,\mathrm{cosec}\,\psi \tag{7b}$$

Equation (6) is a form of Bernoulli's equation that has a closed-form solution.[7] It can be demonstrated that the nonlinear equation can be transformed into a linear equation with an integrating factor through an appropriate variable transformation. The general solution is then given by

$$v(\psi)^{-2} = \kappa\exp\left(-2\int f_1\,d\psi\right) - 2\exp\left(-2\int f_1 d\psi\right) \times\left[\int \exp\left(2\int f_1\,d\psi\right) f_2\,d\psi\right] \tag{8}$$

Evaluating the integrals and determining the constant of integration κ from the boundary conditions, the solution is found to be

$$v(\psi)^{-2} = \left[\frac{\cos(\psi/2)}{\cos(\psi_0/2)}\right]^{2(1+\alpha)}\left[\frac{\sin(\psi/2)}{\sin(\psi_0/2)}\right]^{2(1-\alpha)} \times\left[v_0^{-2} + \frac{\beta\cos^2(\psi_0/2)}{\alpha-1} + \frac{\beta\sin^2(\psi_0/2)}{\alpha+1} + \frac{2\beta\cos^2(\psi_0/2)\sin^2(\psi_0/2)}{\alpha-\alpha^3}\right] - \frac{\beta\cos^2(\psi/2)}{\alpha-1} - \frac{\beta\sin^2(\psi/2)}{\alpha+1} - \frac{2\beta\cos^2(\psi/2)\sin^2(\psi/2)}{\alpha-\alpha^3} \tag{9}$$

which clearly reduces to the vacuum case defined by Eq. (3) as $\beta \to 0$.

The velocity-flight-path-angle profile is shown in Fig. 2 for a range of drag parameters β. It can be seen that as the effect of air drag increases, the trajectory curves more quickly toward the local vertical. Because the effect of drag is to increase the effective thrust-weight ratio of the vehicle, the descent maneuver may be completed with a lower thrust-induced acceleration than would be required for the vacuum descent case.

Conclusions

It has been demonstrated that the conventional solution for a gravity turn maneuver in vacuum may be extended to include descent to the surface of a body with an atmosphere. With the assumption of quadratic air drag, the resulting equations of motion are shown to be a form of Bernoulli's equation, which has a closed analytical solution. With the vehicle velocity available as a function of flight-path angle, the solution for the altitude and time variables is then reduced, in principle, to a set of quadrature integrations.

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Fuel/Time Optimal Control of Spacecraft Maneuvers

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I. Introduction

CONSIDERABLE interest had developed in the study of optimization theory as applied to the spacecraft system by the early 1960s. The basic theory in determining the extrema of the optimal control problems has been developed for nonsingular[1,2] and singular[3] controls. Computation difficulties have plagued the study of time- and fuel-optimal control problems, particularly for systems with nonlinear dynamics. However, there has been a resurgence of interest in the design of controllers for spacecraft reorientation maneuvers in the past decade.[4–10] Among these studies, the optimization objectives have included the maneuver time,[4,5,7] the fuel consumed,[6,8,9] and the weighted fuel/time cost function.[10] In addition, the singular controls of both time- and fuel-optimal controls have been analyzed for spacecraft reorientations.[9,11]

This Note addresses the problem of designing fuel/time-optimal controllers for spacecraft undergoing rest-to-rest maneuvers. A modified switch time optimization (STO) algorithm[12] is used to solve the problem of reorienting an inertially symmetric spacecraft with weighted fuel/time cost function from an initial state of rest to a final state of rest. In this work, we do not study controls with singular arc and, therefore, assume that the fuel/time-optimal control profile is bang-off-bang.[10,13,14] As the weight on the fuel, α, is increased from zero, it is shown that the number of switches in the control profiles, for an inertially symmetric spacecraft, varies from 5 to 10 to 9. Beyond a specific value of α, it is shown that the eigenaxis control with two switches is the optimum.

II. Problem Formulation

The Euler's rotational equations of motion for an inertially symmetric rigid spacecraft with principal body axes at the center of mass are

$$\dot{\omega} = u \tag{1}$$

where $\omega^T = [\omega_1\ \ \omega_2\ \ \omega_3]$ is the angular velocity vector and $u^T = [u_1\ \ u_2\ \ u_3]$, the control vector, is subject to the constraints

$$-1 \le u_i \le 1 \qquad i = 1, 2, 3 \tag{2}$$

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